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Open-closed duality and double scaling

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ABSTRACT: Nonperturbative terms in the free energy of Chern-Simons gauge theory play a key role in its duality to the closed topological string. We show that these terms are reproduced by performing a double scaling limit near the point where the perturbation expansion diverges. This leads to a derivation of closed string theory from this large-N gauge theory along the lines of noncritical string theories. We comment on the possible relevance of this observation to the derivation of superpotentials of asymptotically free gauge theories and its relation to infrared renormalons.

KEYWORDS: 1/N Expansion, AdS-CFT and dS-CFT Correspondence, Topological Strings, 2D Gravity.

Contents

1. Introduction

Ever since 't Hooft's seminal work [1] it has been known that field theories in the large-N limit become *closed* string theories, with $\frac{1}{N}$ playing the role of string coupling constant. Recent developments have revealed that the emergence of such closed string theories is a manifestation of open-closed duality and several well understood examples of such dualities, e.g. the duality of noncritical string theory in two spacetime dimension with matrix quantum mechanics [2] and the duality of certain topological open string theories with topological closed string theories [3]–[6]. The AdS/CFT correspondence [8] is also of this class, though we do not completely understand the formulation of the bulk string theory in this case.

Such dualities have provided us important information about long standing problems in gauge theories. Specifically, in asymptotically free gauge theories like those with $N = 1$ supersymmetry, nonperturbative contributions to the effective superpotential [9] have a direct connection with the topological version of open-closed duality [10]. In this paper we initiate a slightly different understanding of these nonperturbative terms based on a double scaling limit [12] of the underlying large-N gauge theory.

In a typical large-N matrix model characterized by a coupling constant g, usual perturbation theory has a *finite* radius of convergence at $g = g_c$ [13]. At this point the average number of vertices in a typical Feynman diagram diverges so that the diagram becomes a continuous two dimensional surface. If the singularity has the same location g_c at every genus order in the $\frac{1}{N}$ expansion we can define, by performing a double scaling limit, a continuum non critical closed string theory. Generically the contribution to the free energy at genus g diverges as

$$F_g \sim (g - g_c)^{\chi(1+\frac{1}{2m})} \qquad g \geq 2 \tag{1.1}$$

where $\chi = 2 - 2g$ is the Euler characteristic and the integer m depends on the particular matrix model. This behavior allows a definition of nonperturbative string theory by passing to the double scaling limit

$$\begin{aligned} &N \to \infty \qquad g \to g_c \\ &N(g-g_c)^{-(1+\frac{1}{2m})} = \mu = \text{finite}\,. \end{aligned} \tag{1.2}$$

In this string theory genus g amplitudes depend on μ as μ^{χ} and various quantities are functions of the single parameter μ rather than on N and g separately. The emergence of a single parameter is in fact a hallmark of *noncritical string theory*. This simply means that in this string theory, the constant part of the value of the dilaton is not independent of the values of other backgrounds, as opposed to critical string theory where the dilaton is a modulus and can have an arbitrary value. In the double scaling limit the coupling constant g is renormalized to

$$g_R = \frac{g - g_c}{a^2} \tag{1.3}$$

and the *bare* string coupling constant $\frac{1}{N}$ is renormalized to

$$g_s^R = \frac{a^{-(2+\frac{1}{m})}}{N} \tag{1.4}$$

a has dimensions of length and plays the role of a cutoff on the random surface. The parameter μ defined above is then the dimensionless ratio

$$\frac{g_R^{-(2+\frac{1}{m})}}{g_s^R}\,. \tag{1.5}$$

The emergence of closed string theories in the AdS/CFT correspondence or in topological string theory appears to be quite different at first sight. For example in the AdS_5/CFT_4 correspondence, the open string theory in fact becomes a SU(N) gauge theory characterized by a coupling constant g_{YM}. The dual closed string theory also has two independent parameters : the string coupling which is

$$g_s^{crit} = g_{YM}^2 \tag{1.6}$$

and the AdS scale in string units is,

$$\frac{R}{l_s} \sim (g_{YM}^2 N)^{1/4}\,, \tag{1.7}$$

as appropriate for a critical string theory. Similarly in the simplest topological context, the theory of matrices is a Chern-Simons gauge theory while the dual theory is a closed topological string living on a target space characterized by a parameter t (the complexified area of the S^2 resolution of the conifold in string units, whose imaginary part is the B-field on the S^2) and a string coupling g_s where once again $g_s = g_{YM}^2$ and $t = i g_{YM}^2 N$. In this latter case, the duality is exactly known and *nonperturbative* terms of the Chern Simons free energy play an essential role [3, 7].

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One might wonder one can construct a closed string theory similar to noncritical strings starting from Chern Simons theory, using double scaling limit at the radius of convergence of the *perturbative* expansion In this paper we show that this is indeed true and that the double scaled theory exactly corresponds to the $c = 1$ model ($m = \infty$) at self dual radius. The latter is known to be equivalent to closed topological (B model) strings on a S^3 deformation of the conifold [14]. Furthermore we show that the free energy of the double scaled theory precisely reproduces the *nonperturbative* terms of Chern Simons theory by using level rank duality.

The ability to reproduce nonperturbative terms from the perturbative expansion using double scaling tempts us to conjecture that a similar procedure could be valid for other theories as well. In particular we point out that the structure of perturbation theory for Chern-Simons is exactly that of the Borel transform of perturbation expansion of asymptotically free theories and the point beyond which the perturbation expansion diverges corresponds to the first infrared renormalon singularity. It has been recently argued in [15] that these renormalons are key ingredients in the generation of mass gap in asymptotically free theories. This connection may well lead to ways of computing nonperturbative terms in asymptotically free theories.

2. Chern-Simons free energy and toplogical strings

Open topological string on T^*S^3 is exactly three dimensional Chern Simons theory [16]. In the large-N expansion, the free energy has an expansion

$$F = \sum_{g=0}^{\infty} N^{2-2g} F_g(\lambda)\,, \tag{2.1}$$

where

$$\lambda = g_{YM}^2 N = \frac{2\pi N}{k+N} \tag{2.2}$$

is the 't Hooft coupling and k is the level of the Kac-Moody algebra. The genus g term F_g has a perturbative contribution F_g^p and a non-perturbative contribution F_g^{np} where [3]

$$\begin{aligned}
F_0^p(\lambda) &= 2\sum_{p=2}^{\infty} \frac{\zeta(2p-2)}{2p(2p-1)(2p-2)} \left(\frac{\lambda}{2\pi}\right)^{2p-2} \\
F_1^p(\lambda) &= \sum_{p=1}^{\infty} B_2 \frac{\zeta(2p)}{2p} \left(\frac{\lambda}{2\pi}\right)^{2p} \\
F_g^p(\lambda) &= 2\chi_g \sum_{p=1}^{\infty} \zeta(2g-2+2p) \begin{pmatrix} 2g-3+2p \\ 2p \end{pmatrix} \left(\frac{\lambda}{2\pi}\right)^{2g+2p-2} \qquad g \geq 2
\end{aligned} \tag{2.3}$$

and [7]

$$\begin{aligned}
F_0^{np}(\lambda) &= \frac{1}{2} N^2 \left(\log\ (2\pi i \lambda) - \frac{3}{2}\right) \\
F_1^{np}(\lambda) &= -\frac{1}{12} \log\ N \\
F_g^{np}(\lambda) &= N^{2-2g} \chi_g \qquad g \geq 2\,,
\end{aligned} \tag{2.4}$$

where

$$\chi_g = \frac{B_{2g}}{2g(2g-2)} \tag{2.5}$$

and B_n denote the Bernoulli numbers. These nonperturbative terms arise from the volume of the gauge group [7] which is nontrivial since the prefactor of the kinetic term is corrected by quantum effects. They can be obtained in the large N limit using the asymptotic expansion of the Gamma function. Apart from F_0^{np} these nonperturbative terms are in fact a function of N alone.

The expression for the total free energy is in fact exactly that of A-model topological closed string theory on the S^2 resolved conifold geometry. The string coupling constant g_s is related to the 't Hooft coupling by

$$g_s = \frac{i\lambda}{N} \tag{2.6}$$

while the complexified Kahler paramter t of the S^2 by

$$t = i\lambda\,. \tag{2.7}$$

Then the genus-g nonperturbative contribution becomes

$$F_g^{np}(\lambda) = g_s^{2g-2}\frac{B_{2g}}{2g(2g-2)\ t^{2g-2}} \qquad g \geq 2 \tag{2.8}$$

and similarly for the genus zero and one terms. Thus the closed string theory is defined at fixed g_s and the nonperturbative terms are singular in the limit $t \to 0$, a behavior which is essential in the identification of the gauge theory with the string theory.

Notice that if we define the Chern Simons string coupling by $\lambda_s = \frac{2\pi}{k+N}$ the last equation in (2.4) becomes

$$\lambda_s^{2g-2}\frac{B_{2g}}{2g(2g-2)\lambda^{2g-2}} \tag{2.9}$$

In order to go from this expresion to (2.8) we need to replace λ by $t = i\lambda$ i.e. a Wick rotation in λ. The reason for performing this Wick rotation is just in order to get a positive definite total free energy once we sum over all genus. In fact the Bernouilli numbers contain a factor $(-1)^{g-1}$ that makes the total free energy an alternating sum. A similar phenomena appears in the Penner model [17] where a similar Wick rotation is necessary in order to stablish the connection with the $c = 1$ model at the self dual radius [18].

3. A double scaling limit

We now examine whether a knowledge of the perturbative expansion can be used to define a critical limit where a continuum string theory emerges. The perturbative expansions may

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be summed to yield the expressions

$$
\begin{aligned}
F_0^p &= \sum_{n=1}^{\infty}\left[\frac{1}{2}\nu_n^2 \log\left(\frac{\nu_n}{N}\right) - \frac{3}{4}\nu_n^2 + (n \to -n)\right. \\
&\quad \left. - N^2 \sum_{n=1}^{\infty}\left[\log\left(\frac{2\pi}{\lambda}\right) + \frac{4\pi^2 n^2}{\lambda^2}\left(\log\left(\frac{2\pi}{\lambda}\right) - \frac{3}{2}\right)\right]\right] && (3.1)\\
F_1^p &= -\frac{1}{2}B_2 \sum_{n=1}^{\infty}\left[\log\left(\frac{\nu_n}{N}\right) + (n \to -n)\right] + B_2 \sum_n \log\left(\frac{\lambda}{2\pi N}\right) && (3.2)\\
F_g^p &= \chi_g\left(\frac{\lambda}{2\pi}\right)^{2g-2} \sum_{n=1}^{\infty}[\nu_n^{2-2g} + (n \to -n)] + 2\chi_g\left(\frac{\lambda}{2\pi N}\right)^{2g-2} \zeta(2g-2)\,, && (3.3)
\end{aligned}
$$

where we have defined

$$\nu_n = \frac{2\pi N}{\lambda}\left[\frac{\lambda}{2\pi} - n\right]. \tag{3.4}$$

The Chern-Simons theory coupling λ has a fundamental domain between 0 and 2π. It is clear from the above expressions that the perturbation expansion has a finite radius of convergence 2π. In fact the susceptibility diverges at this point with a characteristic critical exponent. Notice that the divergence of the perturbative series (2.3) is for positive value of the Chern Simons coupling. The reason for this is that only amplitudes with an even number of holes are non vanishing.

Therefore, following the usual procedure we will define a critical double scaling limit by taking

$$\lambda \to 2\pi \qquad \nu_1 = \text{finite}. \tag{3.5}$$

In this limit the nonanalytic terms in the free energy become

$$
\begin{aligned}
F_0^{dc} &= \frac{1}{2}\nu_1^2 \log\left(\nu_1 - \frac{3}{2}\right)\\
F_1^{dc} &= -\frac{1}{12}\log\ \nu_1\\
F_g^{dc} &= \chi_g \nu_1^{2-2g}\,,
\end{aligned} \tag{3.6}
$$

where we have used $B_2 = \frac{1}{6}$. The coefficients are exactly those of the *nonperturbative* contributions in (2.4) with N replaced by ν_1. The string coupling is renormalized from its bare value $\frac{\lambda}{N}$ to the renormalized value g_s^R given by

$$g_s^R = \frac{1}{\nu_1}. \tag{3.7}$$

As is well known this nonanalytic piece is in turn identical to the free energy of the $c = 1$ matrix model at the self-dual radius provided we Wick rotate ν to $\mu = i\nu$ as discussed above.[1]

[1]The connection between the $c = 1$ model at the self dual radius and the singular conifold was done in [14] where μ is the complex modulus.

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The string theory which is obtained by performing a double scaling limit near $\lambda = 2\pi$ is in fact related to the string theory defined using the nonperturbative contribution by level rank duality. This is given by the interchange

$$k \leftrightarrow N \tag{3.8}$$

and maps $\lambda = 0$ to $\lambda = 2\pi$. Therefore one could read out the nonperturbative contributions near $\lambda = 0$ from the double scaled expressions near $\lambda = 2\pi$. It is important to notice that the perturbative expansion (2.3) is perfectly smooth and analytic at $\lambda = 0$, while the non analytic contributions (2.4) are non perturbative and are derived from the contribution to the Chern Simons free energy of the volume of the gauge group. Only after including these non perturbative pieces we recover the closed string picture with the Chern Simons t'Hooft coupling λ related to the size t of the resolved conifold by $t = i\lambda$. In the double scaling approach, however, we derive the closed string directly from perturbation theory. This is possible because the Chern Simons perturbative expansion diverges at $\lambda = 2\pi$. The closed string model that we obtain is the non critical $c = 1$ model at the self dual radius that is known to be equivalent to topological strings on the deformed conifold [14]. The double scaled variable ν and the deformation parameter μ of the deformed conifold are related by $\mu = i\nu$. In other words, double scaling limit leads to closed strings on the deformed conifold i.e. the local mirror version of the topological closed string obtained at $\lambda = 0$. By level rank duality of Chern Simons we can relate both mirror topological closed string versions one corresponding to the large level k limit with finite N and the one obtained by double scaling corresponding to large N and finite k. We would like to stress that the moral of this exercise is to show that potentially we can read the dual closed string version of a gauge theory directly from the divergence structure of the perturbative expansion. In next section we will present some general comments in this direction.

4. Structure of perturbation theory, strings and nonperturbative superpotentials

In the previous section we showed that a double scaling limit of the *perturbative* expansion of Chern Simons theory reproduces the *nonperturbative* contribution to the free energy.

Nonperturbative superpotentials for $\mathcal{N} = 1$ SU(N) super Yang Mills can be directly defined using the nonperturbative piece of the Chern Simons free energy with the same gauge group. In fact by compactifying on T^*S^3 type A open topological strings we get [19, 20] perturbative contributions to the F-term superpotential of type

$$W^p(S) = N_c \sum F_{0,h} h S^{h-1} = N_c \frac{\partial F_0^p(S)}{\partial S}\,, \tag{4.1}$$

where $F_{0,h}$ are open topological string amplitudes at genus zero and with h holes. Using now the equivalence between open topological strings on T^*S^3 and Chern Simons on S^3 one can now define

$$W^{np}(S) = N_c \frac{\partial F_0^{np}(S)}{\partial S}\,, \tag{4.2}$$

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where $F_0^{np}(S)$ is the genus zero non perturbative piece of the Chern Simons free energy for $\lambda = S$, namely

$$F_0^{np}(S) = \frac{1}{2}S^2\left(\log\ S - \frac{3}{2}\right). \tag{4.3}$$

Notice that the non peturbative contribution to the superpotential is defined from $F_0^{np}(\lambda)$ given in (2.4) once we extract the corresponding power g_s^{-2} of the string coupling constant. In other words in this approach S is directly related to the Chern Simons coupling λ or in the closed string version to the size t of the S^2 resolved conifold.

In the previous section we have derived the non perturbative piece of Chern Simons free energy by performing a double scaling limit around the critical coupling defining the radius of convergence of the perturbative expansion. The result

$$F_0^{np} = \frac{1}{2}\nu^2\left(\log\ \nu - \frac{3}{2}\right). \tag{4.4}$$

Exactly as before we can read the non perturbative superpotential for S where now S is related with the double scaled variable ν or in the closed string version with the complex modulus of the deformed conifold. This exactly correspond to the mirror type IIB derivation of the $N = 1$ superpotential [10].[2]

After our previous analysis it is natural to ask ourselves if we can use a double scaling limit to define non perturbative physics for asymptotically free theories. Using t'Hooft's double line notation the free energy of a generic SU(N) gauge theory has the form

$$F = \sum_g g_s^{2g-2}F_g(s)\,, \tag{4.5}$$

where s is the t'Hooft coupling defined in the large N limit and $g_s = \frac{1}{N}$. Generically $F_g(s)$ is divergent even for arbitrarily small s. However it is known that in the large-N limit, the Borel transform of $F_g(s)$, denoted by $F_g^B(\lambda)$ has a finite radius of convergence for $\lambda = \lambda_c$ where λ_c is fixed by the location of the first infrared renormalon singularity. Using the Borel transform we can define

$$F^B(\lambda) = \sum_g g_s^{2g-2}F_g^B(\lambda) \tag{4.6}$$

of the full free energy.

We can now try to perform a double scaling limit $\mu = (\lambda - \lambda_c)^\gamma N$ for some γ that will depend on the concrete theory, and to define a nonperturbative $F_{np}^B(\mu)$ in the manner explained in the previous sections. Of course this strategy will only work if the critical value fixing the convergency radius λ_c is the same at any genus order in the $1/N$ expansion. We

[2]As it is standard in double scaling limit for the $c = 1$ model we have $\nu = \frac{\lambda_R}{g_s^R}$ where $\lambda_R = \frac{\lambda-\lambda_c}{a^2}$ and $g_s^R = \frac{1}{a^2 N}$ for a a worldsheet string scale with units of length and where λ_R is defined in the limit $a = 0$ and $\lambda = \lambda_c$ and g_s^R is defined in the double limit $N = \infty$ and $a = 0$. In terms of λ_R and g_s^R we can write $F_0^{np} = g_s^{-2}\lambda_R^2(log(\frac{\lambda_R}{g_s^R}) - \frac{3}{2})$ and to extract the nonperturbative piece of $N = 1$ super Yang Mills from $\lambda_R^2(log(\frac{\lambda_R}{g_s^R}) - \frac{3}{2})$ using $\lambda_R = S$. The main difference with the previous formal construction is that now both λ_R and g_s^R have dimensions. This on the other hand is quite natural since S is also dimensionfull.

can argue that the part of the perturbative expansion dominated by infrared renormalon singularities satisfy this criteria.In fact in contrast to instanton singularities the infrared renormalon is quite universal and does not depend on the number of diagrams contributing to feynman diagram with some topology at a given order in the perturbation expansion.

The variable λ appearing in the Borel transform can be naturally interpreted as the conjugated variable to the gauge theory coupling i.e. as a glueball operator and the double scaled non perturbative Borel transform $F^B_{np}(\mu)$ as a candidate for the effective infrared low energy physics.

Our observation in the Chern Simons exercise can be rephrased by saying that the Chern Simons free energy is capturating the contribution of infrared renormalons to the Borel transform of the $N = 1$ super Yang Mills free energy. In fact for large number n of holes the genus zero contribution to the free energy of Chern Simons goes like

$$\frac{N^2}{\lambda^2} \sum_n \frac{(n-3)!}{n!} \left(\frac{\lambda}{2\pi}\right)^n . \tag{4.7}$$

If we now consider (4.7) as a Borel transform of some perturbative formal series $F(s)$ in s we get

$$F(s) = N^2 \sum_n \frac{(n-3)!}{n(n-1)} \left(\frac{s}{2\pi}\right)^{n-1} . \tag{4.8}$$

If now we formally think of s as the gauge theory t'Hooft coupling the expansion 4.8 behaves at large n typically as an infrared renormalon contribution. Recall that for a generic perturbative expansion $\sum_n a_n s^n$ in t'Hooft coupling s the infrared renormalon corresponds to $a_n \sim b^n n!$ for some coefficient b fixed by the beta function. Notice that the divergence in the perturbative expansion of the Chern Simons free energy can be interpreted as a manifestation of the renormalon of $F(s)$ defined in (4.8). For $N = 1$ super Yang Mills we find the first renormalon singularity at $\lambda_c = 8\pi^2$ while for the Chern Simons analog we get $\lambda_c = 2\pi$.

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